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Ramsey numbers in octahedron graphs

Heiko Harborth *, Ingrid Mengersen

Diskrete Mathematik, Technische Universität Braunschweig, D-38023 Braunschweig, Germany

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Dedicated to Ralph J. Faudree on the occasion of his 60th birthday

Abstract

The octahedron Ramsey number $r_O = r_O(G_1, \ldots, G_t)$ is introduced as the smallest n such that any t-coloring of the edges of the octahedron graph $O_n = K_{2n} - nK_2$ contains for some i a subgraph G_i of color i. With $r = r(G_1, \ldots, G_t)$ denoting the classical Ramsey number, r_O is between $r/2$ and r. If all G_i's are complete, then $r_O = r$. If all G_i's are certain stars, then $r_O = \lceil r/2 \rceil$. For all G_i with at most four vertices, all values $r_O(G_1, G_2)$ are listed.

1. Introduction

In the study of Ramsey numbers, the host graphs are usually the complete graphs K_n. For given graphs $G_1, G_2, \ldots, G_t$, it is asked for the smallest n, the classical Ramsey number $r(G_1, G_2, \ldots, G_t)$, such that any t-coloring (of the edges) of K_n contains for some i a subgraph G_i of color i. Only a few cases of other sequences of host graphs have been studied, for example, complete bipartite graphs [2,9,11–13] and cube graphs [3,15].

The complete graph K_{n+1} represents the vertex points and edges of the n-dimensional simplex, or tetrahedron, the cube graph Q_n consists of vertex points and edges of the n-dimensional cube. For dimension $n \geqslant 5$, there exists only one additional platonic (regular) n-dimensional polytope, namely the octahedron, which has $2n$ vertex points and $\binom{2n}{2} - n$ edges. For this sequence of graphs $O_n = K_{2n} - nK_2$, we will discuss the octahedron Ramsey number $r_O(G_1, \ldots, G_t)$ which is defined as the smallest n such that any t-coloring (of the edges) of O_n contains for some i a subgraph G_i of color i. The existence of $r_O(G_1, \ldots, G_t)$ follows from the existence of $r(G_1, \ldots, G_t)$.

* Correspondence address. TU Braunschweig, Diskrete Mathematik, Pockelsstrasse 14, 38106 Braunschweig, Germany.

E-mail addresses: h.harborth@tu-bs.de (H. Harborth), i.mengersen@tu-bs.de (I. Mengersen).

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2. Relations to classical Ramsey numbers

Since O_n contains K_n as a subgraph, and since O_n differs from K_{2n} only by n independent edges, it is possible to determine $r_O(\ldots)$ from $r(\ldots)$ if a path P_3 is introduced for an additional $(t+1)$st color.

Theorem 1.

$$r_O(G_1,\ldots,G_t) = \lceil \tfrac{1}{2} r(G_1,\ldots,G_t,P_3)\rceil. \tag{1}$$

Proof. To prove '$\leqslant$', consider any t-coloring of $O_{\lceil r/2\rceil}$ with $r = r(G_1,\ldots,G_t,P_3)$. Then add the missing edges in a $(t+1)$st color. We obtain a $(t+1)$-coloring of K_n with $n \geqslant r$ which does not contain a P_3 of color $t+1$. Thus, G_i of color i occurs for some i in K_n and also in $O_{\lceil r/2\rceil}$ since the two graphs differ only in edges of color $t+1$.

For '$\geqslant$', we consider K_n with $n = 2(\lceil r/2\rceil - 1)$. Since $n < r$, there exists a $(t+1)$-coloring of K_n without a G_i of color i, $1 \leqslant i \leqslant t$, and without a P_3 of color $t+1$. Now, it is possible to delete $n/2$ independent edges such that all edges of color $t+1$ are included. It remains a t-coloring of the edges of $O_{n/2}$ without G_i of color i, $1 \leqslant i \leqslant t$. □

Theorem 1 implies the following bounds for $r_O(G_1,\ldots,G_t)$.

Corollary 1.

$$\lceil \tfrac{1}{2} r(G_1,\ldots,G_t)\rceil \leqslant r_O(G_1,\ldots,G_t) \leqslant r(G_1,\ldots,G_t). \tag{2}$$

Proof. We will prove that

$$r(G_1,\ldots,G_t) \leqslant r = r(G_1,\ldots,G_t,P_3) \leqslant 2r(G_1,\ldots,G_t) - 1. \tag{3}$$

Then (2) follows with Theorem 1. The left inequality of (3) is valid since any critical t-coloring of K_n with $n < r$ is a critical $(t+1)$-coloring, too. For the right inequality of (3) we note that K_{2r-1} contains at most $r-1$ edges of color $t+1$ so that there exists a subgraph K_r with colors $1,2,\ldots,t$ only. □

Now, it may be asked whether all values between the bounds of (2) occur as values of r_O. This 'Intermediate Value Question' will be considered in the next section.

3. Exact values

For complete graphs G_i the upper bound of (2) is attained.

Theorem 2.

$$r_O(K_{n_1},\ldots,K_{n_t}) = r(K_{n_1},\ldots,K_{n_t}). \tag{4}$$

Proof. This follows from the result in [4] that

$$r(K_{n_1},\dots,K_{n_t},T)=1+(m-1)(r(K_{n_1},\dots,K_{n_t})-1),$$

where T is a tree with m vertices. Just use (1) and $T=P_3$. □

Also, the lower bound of (2) is achieved, for example, by certain stars.

Theorem 3.

$$\begin{aligned} r_O(K_{1,n_1},\dots,K_{1,n_t}) &= \left\lceil \frac{1}{2}\left(3-t+\sum_{i=1}^{t} n_i\right)\right\rceil \\ &= \begin{cases} \lceil \frac{1}{2} r(K_{1,n_1},\dots,K_{1,n_t})\rceil & \textit{if the number of even } n_i\textit{'s is odd}, \\ 1+\lceil \frac{1}{2} r(K_{1,n_1},\dots,K_{1,n_t})\rceil & \textit{otherwise}. \end{cases} \end{aligned} \tag{5}$$

Proof. From [5] we have

$$r(K_{1,n_1},\dots,K_{1,n_t})=\varepsilon-t+\sum_{i=1}^{t} n_i, \tag{6}$$

where $\varepsilon=1$ if the number of even n_i's is even and positive, and $\varepsilon=2$ otherwise.

With $P_3=K_{1,2}$ we obtain from (6)

$$r(K_{1,n_1},\dots,K_{1,n_t},P_3)=\varepsilon_1+1-t+\sum_{i=1}^{t} n_i, \tag{7}$$

where $\varepsilon_1=1$ if the number of even n_i's is odd, and $\varepsilon_1=2$ otherwise.

Then (7) substituted in (1) gives the first part of (5), and from (6) the second part of (5) follows. □

Consecutive values for r_O starting from the lower bound of (2) are attained if an appropriate complete graph K_p is chosen together with a set of stars $K_{1,n}$ where the number of even n's is even.

Theorem 4.

$$\begin{aligned} &r_O(K_{1,n_1},\dots,K_{1,n_t},K_p) \\ &\quad = \left\lceil \frac{1}{2}\left(1+(p-1)\left(\varepsilon_1-t+\sum_{i=1}^{t} n_i\right)\right)\right\rceil \\ &\quad = \begin{cases} \lceil \frac{1}{2} r(K_{1,n_1},\dots,K_{1,n_t},K_p)\rceil & \textit{if the number of even } n_i\textit{'s is odd}, \\ \lceil \frac{1}{2} r(K_{1,n_1},\dots,K_{1,n_t},K_p)+\frac{p-1}{2}\rceil & \textit{if all } n_i\textit{'s are odd}, \\ \lceil \frac{1}{2} r(K_{1,n_1},\dots,K_{1,n_t},K_p)\rceil+p-1 & \textit{otherwise}, \end{cases} \end{aligned} \tag{8}$$

where $\varepsilon_1=1$ *if the number of even* n_i's *is odd, and* $\varepsilon_1=2$ *otherwise.*

Proof. The result of [14],

$$r_O(K_{1,n_1},\dots,K_{1,n_t},K_p) = 1 + (p-1)(r(K_{1,n_1},\dots,K_{1,n_t}) - 1), \tag{9}$$

with $P_3 = K_{1,2}$ and with (7) will be used to obtain

$$\begin{aligned} r(K_{1,n_1},\dots,K_{1,n_t},K_p,P_3) &= 1 + (p-1)(r(K_{1,n_1},\dots,K_{1,n_t},K_{1,2}) - 1) \\ &= 1 + (p-1)\left(\varepsilon_1 - t + \sum_{i=1}^{t} n_i\right), \end{aligned} \tag{10}$$

where $\varepsilon_1 = 1$ if the number of even n_i's is odd, and $\varepsilon_1 = 2$ otherwise.

The first part of (8) now follows from (1). Using (6), we may interpret (10) as

$$r(K_{1,n_1},\dots,K_{1,n_t},K_p,P_3)$$
$$= \begin{cases} (p-1)(r(K_{1,n_1},\dots,K_{1,n_t}) - 1) + 1 & \text{if the number of even } n_i\text{'s is odd,} \\ (p-1)(r(K_{1,n_1},\dots,K_{1,n_t}) - 1) + p & \text{if all } n_i\text{'s are odd,} \\ (p-1)(r(K_{1,n_1},\dots,K_{1,n_t}) - 1) + 2p - 1 & \text{otherwise.} \end{cases}$$

The second part of (8) follows from this and (9). □

In the simple case of $r_O(G,K_2)$, we may choose G such that any value of the interval of (2) is attained. Let $\beta_1(G)$ denote the edge independence number of G, and let $\bar{G}$ denote the complement of G.

Theorem 5. *If* $|G| = n$, *then*

$$r_O(G,K_2) = n - \beta_1(\bar{G}) = r(G,K_2) - \beta_1(\bar{G}). \tag{11}$$

Proof. With $r(G,K_2,P_3) = r(G,P_3)$ and the result from [6]

$$r(G,P_3) = \begin{cases} n, & \text{if } \bar{G} \text{ has a 1-factor,} \\ 2n - \beta_1(\bar{G}) - 1 & \text{otherwise,} \end{cases}$$

the result (11) follows with (1) and $r(G,K_2) = n$. □

If $G = K_n - iK_2$, $0 \leqslant i \leqslant n/2$, then we obtain any desired value of $r_O(G,K_2)$.

Another general result follows for stars and matchings, where in most cases the lower bound of (2) occurs.

Theorem 6.

$$r_O(K_{1,n_1},\dots,K_{1,n_t},sK_2)$$
$$= \begin{cases} s & \textit{if } \sum_{i=1}^{t} n_i < s + t - 1, \\ \left\lceil \frac{1}{2}\left(\sum_{i=1}^{t} n_i + s - t + \varepsilon_1\right)\right\rceil & \textit{if } \sum_{i=1}^{t} n_i \geqslant s + t - 1 \textit{ and} \end{cases}$$
$$\varepsilon_1 = 1 \textit{ or } \varepsilon_1 = 2 \textit{ for an odd or even number of even } n_i\text{'s}. \tag{12}$$

Proof. The result from [7],

$$r(K_{1,n_1},\dots,K_{1,n_t},sK_2)=\begin{cases}2s & \text{if } \sum_{i=1}^{t} n_i < s+t,\\ \sum_{i=1}^{t} n_i + s - t + \varepsilon - 1 & \text{if } \sum_{i=1}^{t} n_i \geqslant s+t \text{ and}\\ & \varepsilon = 1 \text{ for an even and positive number}\\ & \text{of even } n_i\text{'s, and } \varepsilon = 2 \text{ otherwise,}\end{cases}$$

together with (1) implies (12). □

For two paths where one is quadratically larger than the other, the lower bound in (2) is attained.

Theorem 7.

$$r_O(P_a,P_b)=\left\lceil \frac{1}{2}\left(a+\left\lfloor \frac{b}{2}\right\rfloor -1\right)\right\rceil = \left\lceil \frac{1}{2} r(P_a,P_b)\right\rceil \quad \textit{if } a \geqslant \frac{7}{2}(b+3)^2. \tag{13}$$

Proof. The result from [10],

$$r(P_a,P_b)=a+\left\lfloor \frac{b}{2}\right\rfloor -1 \quad \text{if } a\geqslant b,$$

and the result from [8],

$$r(P_a,P_b,P_c)=a+\left\lfloor \frac{b}{2}\right\rfloor + \left\lfloor \frac{c}{2}\right\rfloor -2 \quad \text{if } a\geqslant \frac{7}{2}(b+c)^2,$$

together with (1) and $c=3$ give the proof. □

4. Small graphs for two colors

From the tables in [1] we obtain with (1) all values of r_O for graphs with at most four vertices (see Table 1, where the little numbers are the corresponding Ramsey numbers $r(G,H)$).

Summarizing, the octahedron Ramsey number r_O, having the octahedron graphs as the host graphs, is not essentially different from the Ramsey number r for complete (or tetrahedron) graphs as the host graphs. In contrast, for cube graphs as the host graphs, the situation is quite different, when the cube Ramsey number r_Q does not exist for some subgraphs of the cube graph Q_n, and when even the existence of r_Q is in question for other subgraphs of Q_n.

Finally, we conjecture that apart from $r_O(P_3,P_3)=r(P_3,P_3)$ the octahedron Ramsey number $r_O(G_1,\dots,G_t)$ and the classical Ramsey number $r(G_1,\dots,G_t)$ are equal, that is, the upper bound of (2) is attained, only if $G_1,\dots,G_t$ are complete graphs.

Table 1
Values of $r_O(G,H)$ with $r(G,H)$ as indices

$G \setminus H$	$2K_2$	P_3	$K_{1,3}$	P_4	C_4	K_3	K_3+e	K_4-e	K_4
$2K_2$	3_5	2_4	3_5	3_5	3_5	3_5	3_5	3_5	4_6
P_3		3_3	3_5	3_4	3_4	3_5	3_5	4_5	4_7
$K_{1,3}$			4_6	4_5	4_6	5_7	5_7	5_7	7_{10}
P_4				3_5	4_5	4_7	4_7	4_7	5_{10}
C_4					4_6	4_7	4_7	5_7	7_{10}
K_3						6_6	6_7	6_7	9_9
K_3+e							6_7	6_7	9_{10}
K_4-e								6_{10}	9_{10}
K_4									18_{18}

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